ELSEVIER
Journal of Geometry and Physics 44 (2002) 156–170
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

# Geometric structure of "broadly integrable" Hamiltonian systems

Francesco Fassò*, Andrea Giacobbe

*Dipartimento di Matematica Pura e Applicata, Università di Padova, Via G. Belzoni 7, 35131 Padova, Italy*

Received 7 January 2002

**Abstract**

We study the geometry of the fibration in invariant tori of a Hamiltonian system which is integrable in Bogoyavlenskij's "broad sense"—a generalization of the standard cases of Liouville and non-commutative integrability. We show that the structure of such a fibration generalizes that of the standard cases. Firstly, the base manifold has a Poisson structure. Secondly, there is a natural way of arranging the invariant tori which generates a second foliation of the phase space; however, such a foliation is not just the polar to the invariant tori. Finally, under suitable conditions, there is a notion of an "action manifold" with an affine structure. We also study the analogous of the problem of the existence of "global action-angle coordinates" for these systems.


*MSC:* 37J35; 37J15; 37J05

*Subj. Class.:* Dynamical systems

*Keywords:* Hamiltonian systems; Broad integrability; Dual pairs; Non-Poisson actions

## 1. Introduction

Integrability of Hamiltonian systems is usually identified with "Liouville" or "complete" integrability: the system has the maximal number of independent integrals of motion in involution ($d$, if the phase space has dimension $2d$) and their level sets are Lagrangian submanifolds which, if compact, are tori. The local structure of this Lagrangian fibration is described by the Liouville–Arnold theorem on action-angle coordinates.

Nevertheless, complete integrability does not exhaust the variety of possible situations. The most important reason is that many systems have more integrals of motion than degrees

* Corresponding author.
*E-mail addresses:* fasso@math.unipd.it (F. Fassò), giacobbe@math.unipd.it (A. Giacobbe).

0393-0440/02/$ – see front matter 
PII: S0393-0440(02)00059-1

of freedom, say $2d-n$ integrals of motion for some $0 < n < d$, and quasi-periodic motions on tori of dimension $n < d$. In many cases, including such important classical systems like the Kepler system and the Euler–Poinsot rigid body, systems of this type are integrable in the “non-commutative” sense of Fomenko and Mischenko [18] (see also [17]).

Geometrically, non-commutative integrability is characterized by the fact that the invariant tori are isotropic and that, moreover, the fibration by the invariant tori is “symplectically complete”. The latter property can be characterized in two equivalent ways: (i) the fibration possesses a “polar” foliation, that is a foliation whose leaves are symplectically orthogonal to the invariant tori, or, (ii) the fibration is a Poisson morphism onto a Poisson manifold $P$. These two characterizations are linked by the fact that the (coisotropic) polar foliation is the lift to the manifold of the symplectic foliation of $P$. This doubly foliated structure, which in symplectic geometry is called with various names, among which “dual pair” [23] and “bifoliation” [16], plays an important role for the comprehension of superintegrable systems (see, e.g. [10] and, for applications, [9,12]).

On the other hand, there are also Hamiltonian systems which have fewer than $d$ integrals of motion but nevertheless have quasi-periodic motions on tori which have dimension greater than $d$ and are coisotropic. Systems of this type have been extensively studied by Parasyuk (see, e.g. [20–22]).

However, the picture is not yet complete because there exist Hamiltonian systems whose motions are quasi-periodic on tori which are neither Lagrangian nor isotropic nor coisotropic [3,4,11]. Even though there is no known example of a system of this kind arising from a mechanical problem, and moreover systems of this kind are rather special (they exist, for instance, only in non-exact symplectic manifolds, see [5,11] for details), their very existence raises some fundamental questions about the concept itself of integrable Hamiltonian system. One further reason of interest is that systems of this kind are related to non-Poisson symplectic torus actions on symplectic manifolds, a subject the study of which has been initiated only very recently (see [2,14,15]).

An important advancement in the comprehension of the situation is due to Bogoyavlenskij [3–5], who formulated a criterion for integrability of Hamiltonian systems (as well as not Hamiltonian systems) which is general enough to account for all the known cases—and to unify them. In short, Bogoyavlenskij calls a Hamiltonian system with $d$ degrees of freedom “integrable in the broad sense” if it possesses

- $2d-n$ first integrals with compact level sets, and
- $n$ independent symmetries which commute and preserve the first integrals.

The common level sets of these integrals of motion, if compact, are obviously $n$-dimensional tori which carry linear motions. Under a natural (but crucial) additional non-resonance hypothesis, called $\mathbb{T}^n$-density, Bogoyavlenskij characterized the local structure of the fibration by these tori in terms of the existence of local coordinates which generalize the action-angle coordinates.

The aim of this paper is to go one step further by investigating the global structure of the fibration by the invariant tori of a $\mathbb{T}^n$-dense broadly integrable Hamiltonian system. Our idea is to try to understand if these systems are fundamentally different from the standard ones by looking at the global properties of the fibration by their invariant tori. As we will see, even though there are some differences, the global geometry of the fibration by the invariant tori generalizes in a natural way the "dual pair" structure of non-commutative integrability. In fact, there is still a naturally defined doubly foliated structure, the second foliation being now the direct sum of the polar foliation and of the fibration itself. Furthermore, as in the standard case, the base of the fibration by the invariant tori has a Poisson structure and the second foliation is the lift of the symplectic foliation of this Poisson manifold. Finally, if the second foliation is a fibration, then as in the standard case its base has an affine structure.

In addition, we will also investigate the analog of the problem of the existence of "global action-angle coordinates" for a broadly integrable system. This problem has been studied by Duistermaat [7] for the completely integrable case, by Dazord and Delzant [8] for the non-commutatively integrable case, and by Parasyuk [21] for the coisotropic case. As we shall point out, the main novelty with respect to the completely and non-commutatively integrable cases is that the monodromy of the fibration and the holonomy of the action manifold do not coincide, even though they are still related.

## 2. Bogoyavlenskij's broad integrability

*Broad integrability*. Bogoyavlenskij's definition of broad integrability applies to any vector field, not necessarily Hamiltonian:

**Definition 1.** A vector field $X$ on a manifold $M$ of dimension $N$ is "integrable in the broad sense" if there exist

(i) A fibration $\pi : M \to P$ with fibers of dimension $n < N$ which are compact, connected and invariant under the flow of $X$. ($P$ is a manifold of dimension $N - n$.)
(ii) For any point $p \in P$, a free infinitesimal action of $\mathbb{T}^n$ on $\pi^{-1}(U)$, where $U \subset P$ is a neighborhood of $p$, which leaves invariant the vector field $X$ and the fibers of $\pi$.

In fact, Bogoyavlenskij expresses condition (i) by requiring the existence of $N - n$ first integrals of $X$ which are almost everywhere independent. The level sets of these integrals give rise to a foliation of $M$. Since we are only interested to the regular leaves of such a foliation and since a submersion with compact fibers is a fibration, we consider a fibration instead. This slightly more general hypothesis allows us to avoid any reference to a particular set of first integrals and to focus on the geometric object itself. Condition (ii) is equivalent to the existence of sets of $n$ "semilocal" vector fields $Y_1, \ldots, Y_n$ which pairwise commute (i.e. $[Y_i, Y_j] = 0$), are infinitesimal symmetries of $X$ (i.e. $[X, Y_i] = 0$) and are tangent to the fibers of $\pi$ (i.e. $L_{Y_i} f = 0$ for any first integral $f$ of $\pi$; a first integral of a fibration is any function constant on its fibers). It is a standard matter to prove that

**Proposition 1.** *If $X$ is integrable in the broad sense then*

(i) *The fibers of $\pi$ are diffeomorphic to $\mathbb{T}^n$.*
(ii) *Every fiber of $\pi$ has a neighborhood $U \subset M$ and a diffeomorphism $\mathcal{C} = (b, \alpha) : U \to W \times \mathbb{T}^n$, where $W \subset \mathbb{R}^{N-n}$, such that*

$$\mathcal{C}^* X|_U(b, \alpha) = w(b)\frac{\partial}{\partial\alpha} \tag{2.1}$$

*with $w$ a map from $W$ to $\mathbb{R}^n$.*

**Proof.** (i) The fibers of $\pi$ are tori because they possess $n$ pairwise commuting and linearly independent tangent vector fields. (ii) Any set of $n$ independent local first integrals of $\pi$ can be taken as $b$-coordinates; the angles $\alpha$ are in fact constructed within the proof of (i). □

In the sequel, we shall often call "invariant tori" the fibers of $\pi$. Our analysis will be restricted to a special (but quite natural) class of broadly integrable systems. The reason for such a restriction is that all the integrals of motion of a system should be taken into consideration when defining the fibration $\pi$—it should not be possible to subdivide all the invariant tori into invariant tori of a smaller dimension. This happens if there are sufficiently many non-resonant tori:

**Definition 2.** A broadly integrable system is said to be "$\mathbb{T}^n$-dense" if a dense set of its invariant tori are closure of orbits.[1]

**Remark.** Following for instance [13], one might call a system "integrable" if it is possible to choose, at least locally, coordinates $(b, \alpha) \in \mathbb{R}^{2d-n} \times \mathbb{T}^n$ such that the vector field has the form (2.1). Any integrable system is broadly integrable: $b_1, \dots, b_{2d-n}$ are integrals of motion and $\partial/\partial\alpha_1, \dots, \partial/\partial\alpha_n$ are commuting symmetries which preserve the $b$'s. Thus, the notion of broad integrability includes essentially all systems with quasi-periodic dynamics and is therefore very general. Whether broad integrability is an effective criterion for integrability depends to a large extent on the possibility of finding natural mechanisms to produce, in concrete cases, the torus fibration (namely, the integrals of motions) and the symmetries.

*The Hamiltonian case*. We now specialize to the (locally) Hamiltonian case. We assume that $M$ has dimension $N = 2d$ and carries a symplectic form $\omega$, and that the vector field $X$ is locally Hamiltonian, namely, the 1-form $i_X\omega$ is closed. A "local Hamiltonian" of $X$ is any function $H$, possibly defined in a subset of $M$, such that $i_X\omega = -\mathrm{d}H$. The vector field is said to be Hamiltonian if $i_X\omega$ is exact. For the present analysis it is natural to consider the locally Hamiltonian case.

The basic result of Bogoyavlenskij's analysis, which is central to the present investigation, is the following Proposition, special cases of which were also given in [11,21]. In order to simplify the notation, we understand sums over repeated indices and we tacitly use the following conventions about indexes: $i, j = 1, \dots, n$; $l, m = 1, \dots, k$; $u, v = 1, \dots, (2d-n-k)/2$, with $k$ defined below.

**Proposition 2** (Bogoyavlenskij). *Let $X$ be a locally Hamiltonian, broadly integrable and $\mathbb{T}^n$-dense vector field on a symplectic manifold $(M, \omega)$. Let $\pi : M \to P$ be the fibration by its invariant tori. Then*:

(i) *The restriction of $\omega$ to the fibers of $\pi$ has constant rank $r = n - k$, $0 \le k \le n$.*
(ii) *Every invariant torus has a neighborhood $U$ equipped with coordinates $(a, p, q, \alpha)$ with values in $\mathbb{R}^k \times \mathbb{R}^{(2d-n-k)/2} \times \mathbb{R}^{(2d-n-k)/2} \times \mathbb{T}^n$ such that*

$$\omega|_U = \Xi_{jl}\, \mathrm{d}a_l \wedge \mathrm{d}\alpha_j + \mathrm{d}p_u \wedge \mathrm{d}q_u + \tfrac{1}{2} C_{ij}\, \mathrm{d}\alpha_j \wedge \mathrm{d}\alpha_i \tag{2.2}$$

[1] Such a system is also said "to have $n$-frequencies" [11,17].

*for a constant antisymmetric $n \times n$ matrix $C$ of rank $r$ and a constant $n \times k$* matrix $\Xi$ *of rank $k$ the rows of which form an orthonormal basis for* ker $C$, *that is*

$$\Xi^{\mathrm{T}} \Xi = \mathbb{I}_k, \qquad \mathrm{Im}\, \Xi = \ker C, \tag{2.3}$$

*where $\mathbb{I}_k$ is the $k \times k$ identity matrix. (If $k = 0$ then $C$ has full rank, there are no $a$-coordinates and $\Xi = 0$. If $k + n = 2d$ then there are no $(p, q)$-coordinates.)*

(iii) *Given a system of such coordinates, there exist a function $h(a)$, unique up to an additive constant, and a unique vector $\eta \in$* Im $C$ *such that*

$$X|_U = \left[ \Xi \frac{\partial h}{\partial a}(a) + \eta \right]_i \frac{\partial}{\partial \alpha_i}. \tag{2.4}$$

*Moreover, every local Hamiltonian of $X$ has the form $h(a) + C\alpha \cdot \eta + const$.*

In the sequel we call "admissible coordinates of action-angle type" or simply "admissible coordinates" any set of semilocal coordinates $(a, p, q, \alpha)$ as in Proposition 2. The coordinates $a$ will be called "actions" and the coordinates $\alpha$ will be called "angles". The meaning of the remaining coordinates $p, q$ will become clear later. Note that the rank of $C$, namely $r = n - k$, is always even.

It may be useful to illustrate the notion of broad integrability giving some examples and remarks. First of all, note that the invariant tori (i.e. the fibers of $\pi$) are isotropic or Lagrangian if and only if $C = 0$, or equivalently $k = n$. In fact, the standard cases of integrability, namely complete and non-commutative integrability, are recovered when $k = n = d$ and, respectively, $k = n < d$. (In the latter case, the existence of the polar foliation follows from the existence of $n$ first integrals of $\pi$ which pairwise Poisson commute, namely, the actions; see [11] for some comments.)

Therefore, in this paper we focus on the case $k < n$. In such a case there are fewer actions than angles, the matrix $C$ is non-zero and hence the symplectic form $\omega$ is not exact. Systems of this kind may have both $n \le d$ and $n \ge d$. When $n < d$ the invariant tori are not isotropic (Lagrangian, if $n = d$); for instance, the locally Hamiltonian system defined by

$$M = \mathbb{R}^2 \times \mathbb{T}^2 \ni (p, q, \alpha_1, \alpha_2), \qquad \omega = \mathrm{d}p \wedge \mathrm{d}q + \mathrm{d}\alpha_1 \wedge \mathrm{d}\alpha_2,$$
$$X = \frac{\partial}{\partial \alpha_1} + \sqrt{2} \frac{\partial}{\partial \alpha_2} \tag{2.5}$$

is a $\mathbb{T}^2$-dense broadly integrable system with no actions ($k = 0$ because in this case $C$ is non-singular). When $n \ge d$ the tori may or may not be coisotropic. As it will be clear later, the coisotropic case is met when $k = 2d - n$.

**Remark.**

(i) The vector fields $\partial/\partial\alpha_i$ are $n$ independent locally Hamiltonian commuting symmetries. If $k < n$, however, not all of them are Hamiltonian (not even in a neighborhood of an invariant torus). In fact, in the coordinate neighborhood $U$, there exist at most $k$ independent commuting symmetries which are Hamiltonian. To see this, denote by $\xi^1, \ldots, \xi^n$ the rows of the matrix $\Xi$, which form an orthonormal basis of ker $C$, and consider any basis $\eta^1, \dots, \eta^{n-k}$ of $(\ker C)^\perp$. Then, the $k$ symmetries $\xi^l \cdot \partial/\partial\alpha$ are Hamiltonian (in $U$) but the $n-k$ symmetries $\eta^j \cdot \partial/\partial\alpha$ are only locally Hamiltonian: one has in fact

$$\omega\left(\xi^l \cdot \frac{\partial}{\partial\alpha}, \cdot\right) = -\mathrm{d}a_l, \qquad \omega\left(\eta^j \cdot \frac{\partial}{\partial\alpha}, \cdot\right) = C\eta^j \cdot \mathrm{d}\alpha.$$

Moreover, when $k < n$ the semilocal $\mathbb{T}^n$-action generating the invariant tori is never Poisson (an action is Poisson if $\omega$ vanishes on any two fundamental vector fields, see, e.g. [19]).

(ii) The flows of the Hamiltonian vector fields $X_{a_1}, \dots, X_{a_k}$ of the actions generate a toral subbundle of the fibration $\pi$ if and only if $\ker C$ is a rational subspace of $\mathbb{R}^n$ (otherwise some of these flows are dense in a torus of dimension greater than $k$).

(iii) In the isotropic case $k = n$ one can always choose the coordinates $a$ so that $\Xi = \mathbb{I}_n$. (In fact $\Xi$ is invertible and one can make the change of coordinates $a \mapsto \Xi a$.)

## 3. Global structure

*The transition functions.* We now study the (global) geometry of the fibration by the invariant tori of a broadly integrable Hamiltonian system. To this end, we begin by determining the transition functions between different sets of semilocal coordinates $(a, p, q, \alpha)$ of Proposition 2.

**Proposition 3.** *Assume $(a, p, q, \alpha)$ and $(a', p', q', \alpha')$ are two systems of semilocal coordinates as in Proposition 2. Denote by $\mathrm{SL}_\pm(n, \mathbb{Z})$ the subgroup of $\mathrm{GL}(n, \mathbb{Z})$ of unimodular matrices and by $\Xi$ and $\Xi'$ the matrices entering the expression (2.2) of $\omega$ in the two coordinate systems. Then, in the intersection of their domains (if not empty) one has*

$$\alpha' = Z\alpha + \mathcal{F}(a, p, q), \tag{3.1}$$

$$a' = \tilde{Z}a + z, \tag{3.2}$$

$$p' = \mathcal{P}(a, p, q), \qquad q' = \mathcal{Q}(a, p, q), \tag{3.3}$$

*where $Z \in \mathrm{SL}_\pm(n, \mathbb{Z})$ is a constant matrix, $z \in \mathbb{R}^k$ is a constant vector, $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{Q}$ are maps, and*

$$\tilde{Z} = \Xi'^{\mathrm{T}} Z^{-\mathrm{T}} \Xi \in \mathrm{GL}(k, \mathbb{R}). \tag{3.4}$$

**Proof.** Let us write $b = (a, p, q)$ and $b' = (a', p', q')$. Thus obviously $b' = b'(b)$. We begin by showing that transformation rule (3.1) for the angles follows from the assumption of $\mathbb{T}^n$-density alone. In fact, let $\alpha' = \alpha'(b, \alpha)$ be such a transformation. Then, along a motion of the system one has[2]

$$\alpha'(b, \alpha_0 + w(b)t) = \alpha'(b, \alpha_0) + w'(b'(b))t.$$

[2] It is understood that all equalities among angles are mod $2\pi$.

For a dense subset of values of $b$, the curve $\alpha_0 + w(b)t$ is dense on the torus; hence, by continuity, for any $b$ one has

$$\alpha'(b, \alpha_0 + \alpha_1) = \alpha'(b, \alpha_0) + \alpha'(b, \alpha_1) \quad \forall \alpha_0, \alpha_1 \in \mathbb{T}^n$$

so that $\alpha \mapsto \alpha'(b, \alpha)$ is linear on each torus: $\alpha'(b, \alpha) = Z(b)\alpha + \mathcal{F}(b)$. Here the matrix $Z(b)$ must belong to $\mathrm{SL}_{\pm}(n, \mathbb{Z})$ and hence is constant.

We now prove the transformation rule (3.2) for the actions (if $k \neq 0$). To this end, we use the simplecticity of the two sets of coordinates, namely

$$\begin{aligned} &\Xi' \, \mathrm{d}a' \wedge \mathrm{d}\alpha' + \mathrm{d}p' \wedge \mathrm{d}q' + \tfrac{1}{2} C' \, \mathrm{d}\alpha' \wedge \mathrm{d}\alpha' \\ &\quad = \Xi \, \mathrm{d}a \wedge \mathrm{d}\alpha + \mathrm{d}p \wedge \mathrm{d}q + \tfrac{1}{2} C \, \mathrm{d}\alpha \wedge \mathrm{d}\alpha. \end{aligned} \tag{3.5}$$

Using $\alpha' = Z\alpha + \mathcal{F}(b)$ one sees that the coefficients of $\mathrm{d}a \wedge \mathrm{d}\alpha$ in the above equation give

$$Z^{\mathrm{T}} \left( \Xi' \frac{\partial a'}{\partial a} + C' \frac{\partial \mathcal{F}}{\partial a} \right) = \Xi. \tag{3.6}$$

Since $\Xi^{\mathrm{T}} \Xi = \mathbb{I}_k$ and $\Xi'^{\mathrm{T}} C' = -(C' \Xi')^{\mathrm{T}} = 0$, see (2.3), one obtains $\partial a' / \partial a = \tilde{Z}$ after left multiplication by $\Xi'^{\mathrm{T}} Z^{-\mathrm{T}}$. In order to conclude the proof it suffices to show that $\partial a'/\partial p = \partial a'/\partial q = 0$. The coefficients of the terms $\mathrm{d}p \wedge \mathrm{d}\alpha$ in (3.5) give $Z^{\mathrm{T}}(\Xi' \partial a'/\partial p + C' \partial \mathcal{F}/\partial p) = 0$, that is

$$\Xi' \frac{\partial a'}{\partial p} = -C' \frac{\partial \mathcal{F}}{\partial p}. \tag{3.7}$$

Since $\mathrm{Im}\, \Xi'$ and $\mathrm{Im}\, C' = (\ker C)^{\perp}$ are orthogonal, the left- and right-hand sides of the latter equations must both vanish. Since $\Xi'$ has maximal rank, this implies $\partial a'/\partial p = 0$, as claimed. The proof that $\partial a'/\partial q = 0$ is analogous. □

**Remark.**

(i) The matrices $C$ and $C'$ satisfy

$$C = Z^{\mathrm{T}} C' Z. \tag{3.8}$$

Eqs. (3.6) and (3.7) imply

$$C' \frac{\partial \mathcal{F}}{\partial a} = Z^{-\mathrm{T}} \Xi - \Xi' \tilde{Z}, \qquad C' \frac{\partial \mathcal{F}}{\partial p} = 0, \qquad C' \frac{\partial \mathcal{F}}{\partial q} = 0. \tag{3.9}$$

(ii) It is not difficult to verify that the functions $h$ and $h'$ and the vectors $\eta$ and $\eta'$ entering the local expressions (2.4) of the vector field $X$ in the two coordinate systems satisfy

$$h(a) = h'(\tilde{Z}a + \tilde{z}) + \beta \cdot \Xi a, \qquad \eta' = Z\eta - Z\beta$$

for some $\beta \in \ker C$. If $X$ is Hamiltonian, then $\beta = 0$.

*The generalized dual pair*. We now use the information of Proposition 3 to infer the global geometry of the fibration $\pi : M \to P$. We assume that $M$ has dimension $2d$, that $P$ has dimension $2d-n$, and that there are $k$ actions. With a minor abuse of notation, the semilocal coordinates $b=(a,p,q)$ can be regarded as local coordinates on the base manifold $P$.

**Proposition 4.**

(i) *$\pi : M \to P$ is symplectically complete, that is, possesses a polar foliation $\pi^\omega$.*

(ii) *$P$ has a Poisson structure of constant rank $2d-n-k$ such that $\pi$ is a Poisson morphism. In local coordinates $(a,p,q)$, the Poisson brackets on $P$ are*

$$\{f,g\}_P = \frac{\partial f}{\partial p_u}\frac{\partial g}{\partial q_u} - \frac{\partial f}{\partial q_u}\frac{\partial g}{\partial p_u},$$

*and $a_1,\dots,a_k$ are local Casimirs.*

**Proof.**

(i) We need to show that the distribution which is orthogonally symplectic to $\pi$ is Frobenius integrable. A vector field $Y$ is symplectically orthogonal to the fibers of $\pi$ if and only if $\omega(Y,\partial/\partial\alpha_i)=0$ for all $i=1,\dots,n$. If we write $Y=Y_{a_l}(\partial/\partial a_l)+Y_{p_u}(\partial/\partial p_u)+Y_{q_u}(\partial/\partial q_u)+Y_{\alpha_i}(\partial/\partial\alpha_i)$ and use the expression (2.2) of $\omega$, this condition becomes

$$\Xi Y_a + C Y_\alpha = 0.$$

As already noticed in the proof of Proposition 3 (see (3.7)) this condition implies $Y_a=0$. Thus, the vector fields symplectically orthogonal to $\pi$ have the form

$$Y = Y_{p_u}\frac{\partial}{\partial p_u} + Y_{q_u}\frac{\partial}{\partial q_u} + Y_{\alpha_i}\frac{\partial}{\partial \alpha_i} \quad \text{with } \; Y_\alpha \in \ker C, \tag{3.10}$$

and with arbitrary $Y_p$ and $Y_q$. Hence, integrability of the distribution amounts to the fact that, if $Y$ and $Y'$ are any two vector fields of the form (3.10), then the $a$-component of the Lie derivative $[Y,Y']$ vanishes and its $\alpha$-component belongs to $\ker C$. The first condition is obviously verified because $Y_a=Y'_a=0$. The second condition is proven by observing that both $Y_\alpha$ and $Y'_\alpha$ are linear combinations of the vectors $\xi_1,\dots,\xi_k$, which form a basis of $\ker C$. Since these vectors are constant, $[Y,Y']_\alpha$ is also a linear combination of them, and hence belongs to $\ker C$.

(ii) It is known that $\pi : M\to P$ is symplectically complete if and only if $P$ is a Poisson manifold and $\pi$ a Poisson morphism [19]. Hence, if $\{\,,\,\}_P$ denote the Poisson brackets of $P$, for any two functions $f$ and $g$ on $P$ one has

$$\{f,g\}_P\circ\pi = \{f\circ\pi, g\circ\pi\}_M = \omega(X_{f\circ\pi}, X_{g\circ\pi}).$$

On the other hand, if $\pi : M\to P$ is symplectically complete, then a function on $M$ is constant on the fibers of $\pi$ if and only if its Hamiltonian vector field belongs to the distribution symplectically orthogonal to $\pi$ [19]. Hence, using the semilocal expression (2.2) of the symplectic form and the fact that $X_{f\circ\pi}$ and $X_{g\circ\pi}$ have the form (3.10), it is immediate to verify that the $\alpha$-components of these two vector fields do not contribute to $\omega(X_{f\circ\pi}, X_{g\circ\pi})$. The conclusion is reached by observing that, on account of (2.2), the $q$ components of $X_f$ are $\partial f/\partial p$, etc. The fact that the $a$'s are Casimirs is obvious. □

**Remark.** Eq. (3.10) shows that a vector field $Y$ symplectically orthogonal to $\pi$ is tangent to $\pi$ if and only if $Y_p = Y_q = 0$. Thus, the invariant tori are coisotropic if and only if there are no $(p, q)$-coordinates, namely, as already stated, if $2d = k + n$.

As we have already mentioned in Section 1, the polar foliation $\pi^\omega$ plays an important role in the case of non-commutatively integrable systems. In the case of a generic broadly integrable system, however, such a foliation need not be invariant under the flow of the system (see remark below) and does not seem to have any special dynamical meaning. Instead, what acquires a dynamical meaning is the foliation obtained by lifting the symplectic foliation of the Poisson manifold $P$ to $M$. Locally, this foliation is defined by $a_1, \dots, a_k =$ const.

**Proposition 5.**

(i) *The lift to M of the symplectic foliation of P coincides with* $\pi + \pi^\omega$. ($\pi + \pi^\omega$ *denotes the foliation generated by the sum of the two distributions generating* $\pi$ *and* $\pi^\omega$.)[3]
(ii) *The leaves of* $\pi + \pi^\omega$ *have dimension* $2d - k$ *and are invariant under the flow of the system*.
(iii) *If the symplectic foliation of P is a fibration* $\sigma : P \to A$, *where A is a k-dimensional manifold, then* $\pi + \pi^\omega$ *is a fibration* $M \to A$ *and A is an affine manifold.*

**Proof.**

(i) The vectors tangent to the foliation described by $a =$ const have the form $P \cdot \partial/\partial p + Q \cdot \partial/\partial q + R \cdot \partial/\partial \alpha$ with $P, Q \in \mathbb{R}^{(2d-n-k)/2}$ and $R \in \mathbb{R}^n$. On the other hand, the vectors tangent to $\pi$ have the form $R \cdot \partial/\partial \alpha$ with $R \in \mathbb{R}^n$ and the vectors tangent to $\pi^\omega$ have the form $P \cdot \partial/\partial p + Q \cdot \partial/\partial q + B \cdot \partial/\partial \alpha$ with $P, Q \in \mathbb{R}^{(2d-n-k)/2}$ and $B \in \ker C$. Thus, the sum of the latter two distributions coincides with the former one, and is therefore Frobenius integrable.
(ii) The foliation is locally defined by $a =$ const, the $a$'s being first integrals of the system.
(iii) This follows from the transition functions (3.2). □

Proposition 5 shows that there are two invariant foliations which are naturally defined in the phase space of a broadly integrable locally Hamiltonian vector field: the fibration $\pi$ by the invariant tori and the foliation $\pi + \pi^\omega$, that we call *action foliation*. If $\pi + \pi^\omega$ is a fibration, then its base manifold $A$ will be called the *action manifold*. $A$ is an affine manifold (even though in general not an affine space) in that it possesses an atlas with affine transition functions

$$a_\lambda = \tilde{Z}_{\lambda\mu} a_\mu + z_{\lambda\mu},$$

this atlas defines an affine (and hence locally flat) connection on $A$, that we call *action connection*.

[3] This implies that if a fibration $\pi$ is symplectically complete then the distribution $\pi + \pi^\omega$ is Frobenius integrable—a statement which can be found in [8].

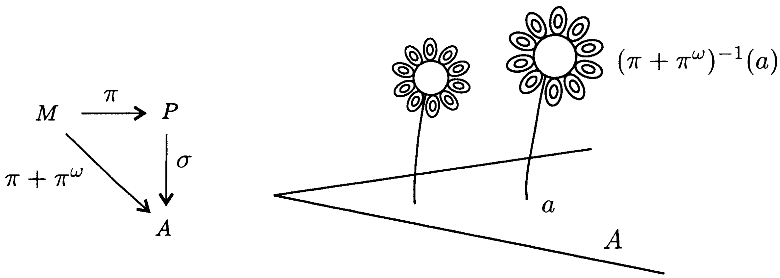


Fig. 1. The structure of the fibration by the invariant tori.

Let us remark that, if the system is integrable in the non-commutative sense, then its invariant tori are isotropic and the leaves of the polar foliation are coisotropic, so that $\pi + \pi^{\omega} = \pi^{\omega}$ and one has an isotropic–coisotropic dual pair. If the system is integrable in the Liouville sense, then the invariant tori are Lagrangian and $\pi + \pi^{\omega} = \pi = \pi^{\omega}$, so the two foliations coincide. The same happens if the system has coisotropic tori, since then $\pi + \pi^{\omega} = \pi$. From this geometric point of view, the coisotropic case closely resembles the Lagrangian one. (The difference is however dynamical, since in the coisotropic case there are necessarily fewer actions than frequencies). The general case resembles the non-commutatively integrable one, with $\pi + \pi^{\omega}$ playing the role of $\pi^{\omega}$.

As in the non-commutatively integrable case, the structure of the fibration by the invariant tori of a broadly integrable system is pictorially representable as in Fig. 1, from [10], where each flower stands for a leaf of the action foliation, its center represents a symplectic leaf, and each petal is an invariant torus. We remark that the picture suggests that each flower is topologically the product $\{\text{symplectic leaf}\} \times \mathbb{T}^n$, but this need not be true since the fibration of each flower by the invariant tori could be topologically non-trivial, see also the next section. Moreover, the picture suggests that the flowers sit on a base manifold $A$, as is the case when the symplectic foliation of $P$ is a fibration.

**Remark .**

(i) As we have anticipated, the polar foliation $\pi^{\omega}$ is in general not invariant under the flow of the system. To show this, let us first observe that the actions $a$ and the local functions $\eta \cdot \alpha$, $\eta \in \operatorname{Im} C$, are first integrals of $\pi^{\omega}$. (In fact, if $Y$ is tangent to $\pi^{\omega}$, then $Y_a = 0$ and $Y_\alpha \in \ker C$ so that $L_Y a_l = 0$ and $L_Y(\eta \cdot \alpha) = 0$.) On account of Proposition 2, the frequencies of motions are $w = \Xi(\partial/\partial a) + \eta$ for some $\eta \in \operatorname{Im} C$, that is, $\eta$ orthogonal to $\operatorname{Im} \Xi$. Thus, $(\mathrm{d}/\mathrm{d}t)\alpha \cdot \eta' = w \cdot \eta' = \eta \cdot \eta'$ is constant for all $\eta' \in \operatorname{Im} C$, but non-zero for some of them.

(ii) The intersection of the tangent spaces to $\pi$ and $\pi^{\omega}$ forms a Frobenius integrable distribution too, but this foliation need not be invariant under the flow of the system and does not seem to have any special dynamical meaning.

## 4. Obstructions to global coordinates

We conclude our investigation by considering the obstructions to the existence of a global system of admissible coordinates of action-angle type, whose local existence is stated in Proposition 2. As already mentioned in Section 1, this investigation extends previous works by Duistermaat [7], Dazord and Delzant [8] and Parasyuk [21]. For general informations on the problem we refer to these works and to [1,6].

To be definite, we consider only the case in which the symplectic foliation of $P$ is a fibration $\sigma : P \to A$. Let us fix an atlas of admissible coordinates of action-angle type

$$(a_\lambda, p_\lambda, q_\lambda, \alpha_\lambda) : U_\lambda \to \mathbb{R}^{2d-n} \times \mathbb{T}^n, \quad \lambda \in \Lambda \subset \mathbb{N}.$$

Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$. Such an atlas allows the definition of four sets of data defined by the coordinate changes (3.1)–(3.3): the matrices $Z_{\lambda\mu}$ and $\tilde{Z}_{\lambda\mu}$, the maps $\mathcal{F}_{\lambda\mu}$, and the vectors $z_{\lambda\mu}$. These families form cocycles, the non-triviality of which constitute obstacles to the existence of global coordinates.

More precisely, the family of matrices $\{Z_{\lambda\mu}\}$ defines a cocycle which is an element of $\surd H^1(\mathcal{U}, \mathrm{SL}_\pm(n, \mathbb{Z}))$ and is called the *monodromy* of the fibration $\pi : M \to P$. The vanishing of this cocycle amounts to the existence of matrices $V_\lambda \in \mathrm{SL}_\pm(n, \mathbb{Z})$, $\lambda \in \Lambda$, such that $Z_{\lambda\mu} = V_\lambda^{-1} V_\mu$ for all $\lambda$ and $\mu$. Equivalently, there is an admissible atlas of action-angle type with all matrices $Z_{\lambda\mu} = \mathbb{I}_n$; this is proven by observing that the new angles $\tilde{\alpha}_\lambda = V_\lambda \alpha_\lambda$ have transition functions $\tilde{\alpha}_\lambda = \tilde{\alpha}_\mu + V_\lambda \mathcal{F}_{\lambda\mu}$.

Similarly, the family $\{\tilde{Z}_{\lambda\mu}\}$ defines a cocycle which is an element of $\surd H^1(\mathcal{U}, \mathrm{GL}(k, \mathbb{R}))$. For reasons to be explained, we call this cocycle the *holonomy* of the action manifold $A$. The vanishing of this cocycle means that $\tilde{Z}_{\lambda\mu} = \tilde{V}_\lambda^{-1} \tilde{V}_\mu$ for certain matrices $\tilde{V}_\lambda, \tilde{V}_\mu \in \mathrm{GL}(k, \mathbb{R})$. Equivalently, there is an admissible atlas of action-angle type with all matrices $\tilde{Z}_{\mu\lambda} = \mathbb{I}_k$. In the case of non-commutative and complete integrability ($k = n$) one has $Z_{\lambda\mu} = \tilde{Z}_{\lambda\mu}$ and the two cocycles coincide, so that there is only one obstruction. The two cocycles have instead different roles in the general case, including the coisotropic one:[4]

**Proposition 6.**

(i) *The monodromy vanishes if and only if $\pi : M \to P$ is a principal bundle.*
(ii) *The holonomy vanishes if and only if the action connection of $A$ has no holonomy (namely, is globally flat).*
(iii) *The holonomy vanishes whenever the monodromy does.*

**Proof.**

(i) As remarked, the monodromy vanishes if and only if there is an admissible atlas for $M$ with angles having transition functions $\tilde{\alpha}_\mu = \tilde{\alpha}_\lambda + V_s \mathcal{F}_{\lambda\mu}$; this is equivalent to the fact that $\pi : M \to P$ is a principal torus bundle.
(ii) If $\tilde{Z}_{\lambda\mu} = (\tilde{V}_\mu)^{-1} \tilde{V}_\lambda$, then the new coordinates $\tilde{a}_\lambda = \tilde{V}_\lambda a_\lambda$ have transition functions $\tilde{a}_\mu = \tilde{a}_\lambda + z_{\lambda\mu}$. Hence the action connection is globally flat.

[4] In [21], the emphasis is on the existence of global coordinates and thus only the monodromy is considered.

(iii) Choose an admissible atlas with all matrices $Z_{\lambda\mu} = \mathbb{I}_n$. Then by (3.8) all matrices $C_\lambda$ are equal to each other, say $C_\lambda = C$, and all matrices $\Xi_\lambda$ have the same image, namely $\ker C$. Thus, the rows of the matrices $\Xi_\lambda$ form different orthonormal bases of $\ker C$. With a linear transformations of the actions $a_\lambda$ in each chart, corresponding to a suitable change of basis in $\ker C$, one can make all matrices $\Xi_\lambda$ equal to each other. Thus, by (3.4), all matrices $\tilde{Z}_{\lambda\mu} = \mathbb{I}_k$. □

**Remark.**

(i) A sufficient condition for the vanishing of the monodromy (respectively, of the holonomy) is the simply connectedness of $P$ (respectively, of $A$).

(ii) In the isotropic case $k = n$, the individual fibers of $\pi + \pi^\omega$ (the flowers of Fig. 1) have no monodromy, in the sense that each of them can be covered with an admissible subatlas with all matrices $Z = \mathbb{I}_n$. Therefore, each of them is a principal torus bundle. It is thus possible to characterize the flowers as families of invariant tori carrying motions with equal frequencies (see [9] for the relevance of this fact in perturbation theory). This is not necessarily true for a generic broadly integrable system.

(iii) When the monodromy is zero $M$ is a symplectic manifold endowed with a non-Poisson principal torus action. Non-Poisson torus actions on compact symplectic manifolds have been studied in [2,14,15]. The compactness hypothesis implies the rationality of $\ker C$; hence, by the Atiyah–Guillemin–Sternberg convexity theorem, the action manifold $A$ is a convex polytope.

(iv) It follows from Proposition 6 and its proof that, if the monodromy vanishes, then there is an admissible atlas such that

$$\begin{aligned} &Z_{\lambda\mu} = \mathbb{I}_n, \qquad \tilde{Z}_{\lambda\mu} = \mathbb{I}_k, \qquad C_{\lambda\mu} = C, \\ &\Xi_{\lambda\mu} = \Xi, \qquad C\,\mathrm{d}\mathcal{F}_{\lambda\mu} = 0 \end{aligned} \tag{4.1}$$

for some matrices $C$ and $\Xi$. The last equality follows from (3.9); in it, $\mathrm{d}\mathcal{F}_{\lambda\mu}$ stands for the Jacobian matrix $(\partial\mathcal{F}_{\lambda\mu}/\partial a, \partial\mathcal{F}_{\lambda\mu}/\partial p, \partial\mathcal{F}_{\lambda\mu}/\partial q)$. Note that, properly speaking, the vanishing of the monodromy does not imply the existence of global action coordinates: on the one hand, the transition functions between the local actions have the form $a_\lambda = a_\mu + z_{\lambda\mu}$ with possibly non-zero $z_{\lambda\mu}$; on the other hand, even if all $z_{\lambda\mu}$ vanish, the local actions $a_\lambda$ define a map $a$ which might be not injective (hence the action manifold need not be diffeomorphic to $\mathbb{R}^k$). However, if the holonomy vanishes, the differentials $\mathrm{d}a_l$ are always global 1-forms.

If the monodromy vanishes, then the families of vectors $z_{\lambda\mu}$ and of maps $\mathcal{F}_{\lambda\mu}$ define other two cocycles. The second of these cocycles is an element of $\sqrt{H^1(\mathcal{U}, \mathbb{T}^n)}$ and is called the *Chern class* of the fibration $\pi : M \to P$. The vanishing of the Chern class means that there exist maps $f_\lambda(a_\lambda, p_\lambda, q_\lambda)$, $\lambda \in \Lambda$, such that

$$\mathcal{F}_{\lambda\mu} = f_\mu - f_\lambda \circ \mathcal{C}_{\lambda\mu},$$

where $\mathcal{C}_{\lambda\mu}$ is the transition map from the $\mu$-coordinates to the $\lambda$-coordinates. From a geometric point of view, the vanishing of the Chern class is equivalent to the existence of a global section of the fibration. In fact, in an admissible atlas with $Z_{\lambda\mu} = \mathbb{I}_n$, the shifted angles $\varphi_\lambda = \alpha_\lambda + f_\lambda$ have transition functions $\varphi_\lambda = \varphi_\mu$, so that the equations $\varphi_\lambda = 0, \lambda \in \Lambda$, define a global section. However, this shift does not produce an admissible system of coordinate of action-angle type, unless the functions $f_\lambda$ have some special property, see Proposition 8.

**Proposition 7.** *Assume that the monodromy vanishes. Then*

(i) *The cocycle $\{z_{\lambda\mu}\}$ vanishes if and only if the Poisson manifold $P$ has $k$ independent global Casimirs.*
(ii) *The Chern class vanishes if and only if the fibration $\pi : M \to P$ is trivial.*
(iii) *If the Chern class vanishes, then there exists a (non-unique) closed 2-form in $P$ that, restricted to the symplectic leaves, is the symplectic structure of the leaves. We call any such 2-form a "Poisson 2-form".*

**Proof.** Choose an admissible atlas which satisfies conditions (4.1).

(i) Triviality of $\{z_{\lambda\mu}\}$ means $z_{\lambda\mu} = v_\mu - v_\lambda$ for some vectors $v_\lambda, v_\mu$. Thus, the components of the shifted actions $a_\lambda + v_\lambda$ have identical transition functions and define $k$ independent global Casimirs.
(ii) A principal bundle which has a global section is trivial.
(iii) Let $\mathcal{F}_{\lambda\mu} = f_\mu - f_\lambda$. We show that the 2-forms defined within each coordinate domain by

$$\tilde{\omega}_\lambda = \Xi\, \mathrm{d}a_\lambda \wedge \mathrm{d}f_\lambda + \mathrm{d}p_\lambda \wedge \mathrm{d}q_\lambda$$

are the local representatives of a closed, non-degenerate 2-form on $P$. Clearly, the only fact that needs to be proven is that the local forms $\tilde{\omega}_\lambda$ are compatible, that is, $\tilde{\omega}_\lambda = \tilde{\omega}_\mu$ on intersecting domains. Since $C\, \mathrm{d}\mathcal{F}_{\lambda\mu} = 0$, one has

$$C\, \mathrm{d}\alpha_\lambda \wedge \mathrm{d}\alpha_\lambda = C(\mathrm{d}\alpha_\mu + \mathrm{d}\mathcal{F}_{\lambda\mu}) \wedge (\mathrm{d}\alpha_\mu + \mathrm{d}\mathcal{F}_{\lambda\mu}) = C\, d\alpha_\mu \wedge \mathrm{d}\alpha_\mu,$$

and hence the symplecticity condition

$$\begin{aligned} &\Xi\, \mathrm{d}a_\lambda \wedge \mathrm{d}\alpha_\lambda + \mathrm{d}p_\lambda \wedge \mathrm{d}q_\lambda + \tfrac{1}{2} C\, d\alpha_\lambda \wedge \mathrm{d}\alpha_\lambda \\ &\quad = \Xi\, da_\mu \wedge \mathrm{d}\alpha_\mu + \mathrm{d}p_\mu \wedge \mathrm{d}q_\mu + \tfrac{1}{2} C\, \mathrm{d}\alpha_\mu \wedge \mathrm{d}\alpha_\mu \end{aligned}$$

reduces to $\tilde{\omega}_\lambda = \tilde{\omega}_\mu$ since $\mathrm{d}a_\lambda = \mathrm{d}a_\mu$. □

In the Lagrangian and isotropic cases, a sufficient condition for the vanishing of the cocycle $\{z_{\lambda\mu}\}$ is the exactness of the symplectic 2-form $\omega$ [7,8]. This condition plays no role in our case since in the non-isotropic case the symplectic form cannot be exact.

Results about the existence of "global action-angle coordinates" have been given in the already quoted articles by Duistermaat, Dazord and Delzant and Parasyuk under the hypothesis that the Chern class vanishes. These results generalize to the following proposition.

**Proposition 8.** *Assume that the monodromy and the Chern class vanish. Consider an admissible atlas which satisfies conditions* (4.1). *Denote $\mathcal{F}_{\lambda\mu} = f_\mu - f_\lambda$ and assume that* *there is a choice of the functions* $f_\lambda$ *such that*

$$C\,\mathrm{d}f_\lambda = 0, \quad \lambda \in \Lambda. \tag{4.2}$$

*Then, there exist global 1-forms* $\mathrm{d}a_l$, *global angles* $\alpha_i$, *and a Poisson 2-form* $\tilde{\omega}$ *such that*

$$\omega = \Xi\,\mathrm{d}a \wedge \mathrm{d}\alpha + \tfrac{1}{2}C\,\mathrm{d}\alpha \wedge \mathrm{d}\alpha + \tilde{\omega}.$$

**Proof.** Define $\alpha = \alpha_\lambda + f_\lambda$ in each chart domain, $\lambda \in \Lambda$. Since $C\,\mathrm{d}f_\lambda = 0$, the local expression of the symplectic form $\omega$ is

$$\Xi\,\mathrm{d}a_\lambda \wedge \mathrm{d}\alpha_\lambda + \mathrm{d}p_\lambda \wedge \mathrm{d}q_\lambda + \tfrac{1}{2}C\,\mathrm{d}\alpha_\lambda \wedge \mathrm{d}\alpha_\lambda = \Xi\,\mathrm{d}a \wedge \mathrm{d}\alpha + \tilde{\omega}_\lambda + \tfrac{1}{2}C\,\mathrm{d}\alpha \wedge \mathrm{d}\alpha,$$

as claimed. □

**Remark.** Condition (4.2) is a compatibility requirement between the Poisson 2-form and the choice of the angle coordinates. This condition obviously does not appear in the Lagrangian and isotropic cases. An equivalent condition (exactness of a certain form $\Theta$) is used by Parasyuk in his study of the coisotropic case. Parasyuk also gives conditions which ensure that $\tilde{\omega}$ is exact: these conditions have some interest in the coisotropic case because, in that case, they ensure that it is possible to eliminate $\tilde{\omega}$ with a change of the angle coordinates.

## Acknowledgements

This work was supported by the EU contract HPRN-CT-2000-0113 for the project MASIE—Mechanics and Symmetry in Europe.

## References

[1] L.M. Bates, Monodromy in the champagne bottle, J. Appl. Math. Phys. (ZAMP) 42 (1991) 837–847.
[2] Y. Benoist, Actions symplectiques de groupes compacts, C.R. Acad. Sci. Paris Ser. I Math. 327 (1998) 373–376.
[3] O.I. Bogoyavlenskij, A concept of integrability of dynamical systems, C.R. Math. Rep. Acad. Sci. Canada 18 (1996) 163–168.
[4] O.I. Bogoyavlenskij, Theory of tensor invariants of integrable Hamiltonian systems. II. Theorem on symmetries and its applications, Commun. Math. Phys. 184 (1997) 301–365.
[5] O.I. Bogoyavlenskij, Extended integrability and bi-Hamiltonian systems, Commun. Math. Phys. 196 (1998) 19–51.
[6] R. Cushman, L. Bates, Global Aspects of Classical Integrable Systems, Birkhauser, Basel, 1997.
[7] J.J. Duistermaat, On global action-angle coordinates, Commun. Pure Appl. Math. 33 (1980) 687–706.
[8] P. Dazord, T. Delzant, Le probleme general des variables actions-angles, J. Diff. Geom. 26 (1987) 223–251.
[9] F. Fassò, Hamiltonian perturbation theory on a manifold, Cel. Mech. Dyn. Astr. 62 (1995) 43–69.
[10] F. Fassò, The Euler–Poinsot top: a non-commutatively integrable system without global action-angle coordinates, J. Appl. Math. Phys. (ZAMP) 47 (1996) 953–976.
[11] F. Fassò, Quasi-periodicity of motions and complete integrability of Hamiltonian systems, Ergodic Theory Dyn. Syst. 18 (1998) 1349–1362.
[12] F. Fassò, T. Ratiu, Compatibility of symplectic structures adapted to noncommutatively integrable systems, J. Geom. Phys. 27 (1998) 199–220.

[13] G. Gallavotti, Quasi-integrable mechanical systems, in: K. Osterwalder, R. Stora (Eds.), Critical Phenomena, Random Systems, Gauge Theories, Les Houches, Session XLIII, 1984, North-Holland, Amsterdam, 1986.
[14] A. Giacobbe, Convexity and multivalued Hamiltonians, Russ. Math. Surveys 55 (2000) 578–580.
[15] A. Giacobbe, Convexity of multivalued momentum maps, Preprint, 2000. http://front.math.ucdavis.edu/math.SG/0004061.
[16] M.V. Karasev, V.P. Maslov, Nonlinear Poisson Brackets. Geometry and Quantization, Translations of the AMS, Vol. 119, AMS, Providence, RI, 1993.
[17] N.N. Nekhoroshev, Action-angle variables and their generalizations, Trans. Moskow Math. Soc. 26 (1972) 180–198.
[18] A.T. Fomenko, A.S. Mishenko, A generalized method for Liouville integration of Hamiltonian systems, Funct. Anal. Appl. 12 (1978) 46–56.
[19] P. Libermann, C.-M. Marle, Symplectic Geometry and Analytical Mechanics, Reidel, Dordrecht, 1987.
[20] I.O. Parasyuk, Co-isotropic invariant tori of Hamiltonian systems in the quasiclassical theory of the motion of a conduction electron, Ukrainian Math. J. 42 (1990) 308–312.
[21] I.O. Parasyuk, Variables of the action-angle type on symplectic manifolds stratified by coisotropic tori, Ukrainian Math. J. 45 (1993) 85–93.
[22] I.O. Parasyuk, Reduction and coisotropic invariant tori of Hamiltonian systems with non-Poisson commutative symmetries. II, Ukrainian Math. J. 46 (1994) 991–1002.
[23] A. Weinstein, The local structure of Poisson manifolds, J. Diff. Geom. 18 (1983) 523–557.